Eur. Phys. J. B **21**, 103–107 (2001)

THE EUROPEAN PHYSICAL JOURNAL B
EDP Sciences

Chaotic solitons in Sine-Gordon system

Wenhua Hai[1,2,a], Zelan Zhang[3], and Jianshu Fang[2]

[1] CCAST (World Laboratory), PO Box 8730, Beijing 100080, PR China
[2] Department of Physics, Hunan Normal University, Changsha 410081, PR China
[3] Department of Analysis and Measurement, Wuhan University, Wuhan 430072, PR China

Received 25 December 2000

Abstract. We extend the constant-variation method to the case of partial differential equations. Applying the method to periodically perturbed Sine-Gordon system, we find some novel solitons, which are embedded in a chaotic attractor and possess controllable velocity of motion. Taking periodically driven long Josephson junction as an example the corresponding chaotic region in parameter space and chaotic orbit are obtained analytically and numerically.

PACS. 05.45.Ac Low-dimensional chaos – 02.60.Cb Numerical simulation; solution of equations – 74.50.+r Proximity effects, weak links, tunneling phenomena, and Josephson effects

Soliton and chaos are two kinds of important and interesting phenomena in nonlinear dynamics and the corresponding quantum mechanics [1–4]. Solitons describe regular motions of some completely integrable systems and chaos means stochastic motion in the presence of non-integrability for the deterministic systems. The regularity and randomness can be used to same field for different purposes, as in the soliton communication [5] and chaos one [6,7]. However, the integrability of soliton systems could easily be broken by the periodic perturbations, which generally exist in really physical situations. It had been shown numerically and semianalytically that the solitons interacting with an external oscillating field could become stochastically unstable one [4,8–10]. The stable solitons embedded in a chaotic attractor would be doubly useful for the practical problems, which require both regularity and randomness. In this paper, we only investigate the soliton and chaos in a Sine-Gordon system with periodical perturbations.

The Sine-Gordon (SG) solitons are the model of many physically interesting problems such as the magnetic-flux propagation on a rf-driven long Josephson-junction [11,12], the charge-density waves in a one-dimensional condensate interacting with an ac electric field [13] and the B-DNA molecular groups experiencing microwaves [14–16]. Recently, we suggest a direct perturbation technique for handling the perturbed SG solitons [17,18] and the homoclinic chaos [19,20] in the geometrically one-dimensional case. All of the results [17–20] show that stabilities of the systems depend on the initial and boundary conditions. We will give a similar conclusion and some new results for the single solitons perturbed by a periodical field here. That is, we will extend the above method to (1+1) dimensional case and use it to obtain the chaotic solitons whose boundedness sensitively depends on the initial conditions and system parameters. Particularly, the boundedness conditions make the soliton velocity controllable, which is a useful property.

Equation describing interaction between the SG solitons and perturbed external field is [8,9]

$$\varphi_{tt} - \varphi_{xx} + \sin\varphi = \varepsilon R(x,t,\varphi) \quad for \quad |\varepsilon| \ll 1 \tag{1}$$

in suitably normalized units, where $R(x,t,\varphi)$ is a periodic function of time t and all of quantities are dimensionless. Making the perturbation expansion

$$\varphi = \varphi^{(0)} + \sum_{i=1}^{\infty} \varepsilon^i \varphi^{(i)}, \tag{2}$$

and inserting it into equation (1) yield the unperturbed equation

$$\varphi_{tt}^{(0)} - \varphi_{xx}^{(0)} + \sin\varphi^{(0)} = 0 \tag{3}$$

and any ith-order perturbed equations

$$\varphi_{tt}^{(i)} - \varphi_{xx}^{(i)} + (\cos\varphi^{(0)})\varphi^{(i)} = R^{(i)}(x,t,\varphi^{(j)}) \quad \text{for} \ \ j < i, \ i = 1,2,... \tag{4}$$

Here number i denotes the order of approximation and $R^{(i)}(x,t,\varphi^{(j)})$ is the ith-order perturbed function from expansion of the function $R(x,t,\varphi)$. For the similar equations with spatially independent φ, we have derived its chaotic solution from the well-known constant-variation and demonstrated sensitivity of the solution to the initial conditions [19,20].

[a] e-mail: adcve@public.cs.hn.cn

Now we extend this result from one-dimension to the considered (1+1) dimension. We take the periodically driven long Josephson junction as an instance, in which the perturbed functions read

$$R(x,t,\varphi) = \eta - \alpha\varphi_t + \rho\sin(kx-\omega t), \quad (5)$$

$$R^{(1)}(x,t,\varphi^{(0)}) = \eta - \alpha\varphi_t^{(0)} + \rho\sin(kx-\omega t), \quad (6)$$

$$R^{(i)}(x,t,\varphi^{(j)}) = -\alpha\varphi_t^{(i-1)} + h^{(i)}(\varphi^{(j)}), \quad i=2,3,..., \quad j<i. \quad (7)$$

Here α denotes the dissipative factor, η is the normalized bias current density, the term proportional to ρ represents an electromagnetic field with the normalized amplitude ρ , wave vector k and frequency ω, and $h^{(i)}(\varphi^{(j)})$ is the function of $\varphi^{(j)}$ which comes from the expansion of $\sin\left(\sum_{i=1}\varepsilon^i\varphi^{(i)}\right)$. Note that space and time have been normalized to the characteristic Josephson length and the inverse plasma frequency here.

It is well known that equation (3) possesses the exact solutions containing both localized multisoliton and periodic processes in the (1+1) dimension case [2,21]. Inserting any one of them into equation (4) yields the first order equation as a non-homogeneous one with the variable coefficient $\cos\varphi^{(0)}$. On the other hand there exists the exact solution of soliton equation (1) when the periodic contribution in equation (5) is not small and $\eta=\alpha=0$ [21]. If we take this type as zero-order solution, then the perturbation would concern only the bias and dumping terms.

The substitution of equations (6, 7) into equation (4) makes the latter the non-homogeneous equations (NHE) with the non-homogeneous term $R^{(i)}(x,t,\varphi^{(j)})$ for $j<i$. At $R^{(i)}=0$, the corresponding homogeneous equations (HE) of equations (4) are in same form for any i. Assuming the HE has the two linearly independent solutions φ_1 and φ_2. Then the linearity of the HE leads to the new solution $C_i\varphi_1+D_i\varphi_2$ with C_i, D_i being arbitrary constants. The idea of constant-variation implies that using functions $f^{(i)}(x,t)$ and $g^{(i)}(x,t)$ instead of the constants C_i and D_i can construct solution of the NHE (4). Setting

$$\varphi^{(i)} = f^{(i)}(x,t)\varphi_1 + g^{(i)}(x,t)\varphi_2 \quad \text{for} \quad i=1,2,... \quad (8)$$

and inserting it into equation (4), we give the coupled equations of the functions $f^{(i)}$ and $g^{(i)}$ as

$$\begin{aligned}&\varphi_1 f_t^{(i)} + \varphi_2 g_t^{(i)} = 0, \quad \varphi_1 f_x^{(i)} + \varphi_2 g_x^{(i)} = 0,\\ &\varphi_{1,t} f_t^{(i)} + \varphi_{2,t} g_t^{(i)} + \varphi_{1,x} f_x^{(i)} + \varphi_{2,x} g_x^{(i)} = R^{(i)}(x,t,\varphi^{(j)}).\end{aligned} \quad (9)$$

Combining equations (8) and (9) with the latter equations (13) produces the separate equations

$$\begin{aligned}f_t^{(i)} - Df_x^{(i)} &= -D^{-1}\sqrt{1-D^2}\varphi_2(x,t)R^{(i)}(x,t,\varphi^{(j)}),\\ g_t^{(i)} - Dg_x^{(i)} &= D^{-1}\sqrt{1-D^2}\varphi_1(x,t)R^{(i)}(x,t,\varphi^{(j)}).\end{aligned} \quad (10)$$

Solutions of equations (10) are mathematically well-known

$$\begin{aligned}f^{(i)} &= -D^{-1}\sqrt{1-D^2}\int_{t_0}^{t}\varphi_2[x'(x,t,\tau),\tau]\\ &\quad\times R^{(i)}\{x'(x,t,\tau),\tau,\varphi^{(j)}[x'(x,t,\tau),\tau]\}\mathrm{d}\tau,\\ g^{(i)} &= D^{-1}\sqrt{1-D^2}\int_{t_0}^{t}\varphi_1[x'(x,t,\tau),\tau]\\ &\quad\times R^{(i)}\{x'(x,t,\tau),\tau,\varphi^{(j)}[x'(x,t,\tau),\tau]\}\mathrm{d}\tau,\end{aligned} \quad (11)$$

for $i>j$, where by $x'=x-D(\tau-t)$ we mean the characteristic line of equations (10). Note that x, t and D are dimensionless quantities. Clearly, the solution (2) with equations (8) and (11) satisfies the usual initial condition $\varphi(t_0)=\varphi^{(0)}(t_0)$ or $\varphi^{(i)}(t_0)=0$ of the perturbed soliton system.

When $i=1$, $j=0$, equations (11) depend on the zero-order solution of equation (3). Let us adopt the simplest single solitons of equation (3) to evidence the above result. The single soliton solutions of equation (3) are

$$\varphi^{(0)} = 4\tan^{-1}\exp\xi, \quad \xi = \pm\frac{1}{\sqrt{1-D^2}}(x-Dt-\xi_0), \quad (12a)$$

$$\xi_0 = x_0 - Dt_0 \pm \sqrt{1-D^2}\ln\tan\left[\varphi^{(0)}(t_0,x_0)/4\right], \quad (12b)$$

where D is dimensionless velocity of the solitons, $\varphi^{(0)}(t_0,x_0)$ is the value of $\varphi^{(0)}$ at the initial time t_0 and boundary coordinate x_0, the positive and negative signs are associated with kink and antikink solitons respectively. These solutions are only valid for an infinite system. In reality, any system is of finite length and is governed by suitable boundary conditions. For a long junction, we can use equations (12) as an good approximation of the real solution. Substituting equations (12) into equation (4) and setting $R^{(i)}=0$ we easily construct two linearly independent solutions

$$\varphi_1 = \mathrm{sech}\xi, \quad \varphi_2 = \frac{1}{2}(\sinh\xi + \xi\mathrm{sech}\xi) \quad (13)$$

of the HE. Applying (x',τ) to replace (x,t) in equations (6) and (13), and inserting them into equations (11) yield the first-order solutions of equations (10) in the form

$$\begin{aligned}f^{(1)} = &-\frac{(1-D^2)}{4D^2}\int_{\zeta_1}^{\zeta_2}(\sinh\xi' + \xi'\mathrm{sech}\xi')\\ &\times\left[\eta \pm \frac{2\alpha D}{\sqrt{1-D^2}}\mathrm{sech}\xi' + \rho\sin(\zeta + a\xi')\right]\mathrm{d}\xi',\end{aligned} \quad (14a)$$

$$\begin{aligned}g^{(1)} = &\frac{(1-D^2)}{2D^2}\int_{\zeta_1}^{\zeta_2}\mathrm{sech}\xi'\\ &\times\left[\eta \pm \frac{2\alpha D}{\sqrt{1-D^2}}\mathrm{sech}\xi' + \rho\sin(\zeta + a\xi')\right]\mathrm{d}\xi',\end{aligned} \quad (14b)$$

where the following representations have been used,

$$a = \frac{1}{2}\Big(k + \frac{\omega}{D}\Big)\sqrt{1-D^2},$$
$$\zeta = \frac{1}{2}\Big(k - \frac{\omega}{D}\Big)(x+Dt) + \frac{1}{2}\Big(k + \frac{\omega}{D}\Big)\xi_0; \qquad (15)$$

$$\xi' = (x + Dt - \xi_0 - 2D\tau)/\sqrt{1-D^2},$$
$$\zeta_1 = \xi'|_{\tau=t_0} = \frac{(x+Dt-\xi_0-2Dt_0)}{\sqrt{1-D^2}},$$
$$\zeta_2 = \xi'|_{\tau=t} = \frac{(x-Dt-\xi_0)}{\sqrt{1-D^2}}\cdot \qquad (16)$$

When $k=0$, the result describes the corresponding problem of the oscillating field [8,9].

Examining the first-order solution (8) for $i=1$ with equations (13) and (14) we find two interesting properties: (a) The solution contains some insolvable integrations, which evidences the non-integrability of the chaotic system. (b) The solution is bounded if and only if the conditions

$$\lim_{t\to\infty} g^{(i)}(x,t) = 0 \quad \text{for} \quad i = 1,2,... \qquad (17)$$

are satisfied. Necessity of the conditions (17) is obvious, because of the unboundedness of φ_2 as t tends to infinity. Also equation (17) makes possible to use l'Hospital rule deriving the finite superior limits of $|\varphi|$ and $|\varphi_t|$, that is proof of the sufficiency [19]. The boundedness of the perturbed solution was usually associated with the Lyapunov stability [22]. Therefore equations (17) are really the conditions for possible stability of single solitons (5) under the deterministic perturbation. Applying equations (14b, 15, 16) to equations (17) results in explicit form of the condition for $i=1$ as

$$\lim{}_{t\to\infty}\Big\{\pi\eta \pm \frac{4\alpha D}{\sqrt{1-D^2}} + \pi\rho\,\mathrm{sech}\Big[\frac{\pi}{4}(k+\frac{\omega}{D})\sqrt{1-D^2}\Big] \times \sin\Big[\frac{1}{2}(k+\frac{\omega}{D})\xi_0 + \frac{1}{2}(k-\frac{\omega}{D})(x+Dt)\Big]\Big\} = 0. \qquad (18)$$

This condition cannot be satisfied for general case, since it contains the variables x and t. Therefore the perturbed solutions (8) are generally unbounded, as in the oscillating field case [8,9] with $k=0$. In order to make existence of the bounded solitons, the soliton velocity D must obey the dispersion relation

$$D = \omega/k. \qquad (19)$$

Under this relation using equation (12b) to the condition (18) yields

$$\pi\eta \pm \frac{4\alpha\omega}{\sqrt{k^2-\omega^2}} + \pi\rho\,\mathrm{sech}\Big[\frac{\pi}{2}\sqrt{k^2-\omega^2}\Big]\sin(k\xi_0) = 0,$$
$$\xi_0 = x_0 - Dt_0 \pm \sqrt{1-D^2}\ln\tan[\varphi^{(0)}(t_0,x_0)/4]. \qquad (20)$$

Thus we have demonstrated that only for the parameters determined by equations (19) and (20) the solution (8) are bounded.

Let us look at an example of the chaotic soliton. Applying equations (19) and (12) with positive sign to equation (6) produces the first-order perturbed function

$$R^{(1)} = \eta + \frac{2\alpha\omega}{\sqrt{k^2-\omega^2}}\mathrm{sech}\xi + \rho\sin(\sqrt{k^2-\omega^2}\xi + k\xi_0). \qquad (21)$$

Inserting equations (21) and (13) into equation (11) can give the (1+1) dimensional chaotic soliton. For simplicity here we only seek travelling wave solution $f = f(\xi)$, $g = g(\xi)$ of equations (10) such that the equations become

$$f_\xi^{(1)} = \frac{k^2-\omega^2}{2\omega^2}\varphi_2(\xi)R^{(1)}(\xi),$$
$$g_\xi^{(1)} = -\frac{k^2-\omega^2}{2\omega^2}\varphi_1(\xi)R^{(1)}(\xi) \qquad (22)$$

for $i=1$. Integrating the two equations and inserting them into equations (8) yields the first-order chaotic soliton solution

$$\varphi^{(1)} = \frac{k^2-\omega^2}{2\omega^2} \times \Big[\varphi_1(\xi)\int_A^\xi \varphi_2(\xi)R^{(1)}(\xi)\mathrm{d}\xi - \varphi_2(\xi)\int_B^\xi \varphi_1(\xi)R^{(1)}(\xi)\mathrm{d}\xi\Big], \qquad (23)$$

where A and B are integration constants. Setting the parameter set $k=\sqrt{2}$, $\omega=1$, $\eta=\rho=0.1$ and $k\xi_0=\pi/2$, equation (20) gets the damping $\alpha = 0.025\pi[1+\mathrm{sech}(\pi/2)]$. Given the parameters and selected the constants $A=B=0$, we substitute equations (13) and (21) into equation (23) to plot ξ space-time evolution as Figure 1a, by using the "Mathematica". The corresponding phase orbit $\varphi^{(1)}$ *versus* $\varphi_\xi^{(1)}$ is drawn in Figure 1b. These plots show that the ξ space-time evolution and phase orbit of the chaotic soliton are very complex. Particularly, the numerical solution tends to unboundedness for sufficiently large value of the variable ξ, although the corresponding analytical solution (23) is bounded under the condition (17). This exhibits that sensitivity of the solution to the initial conditions leads to spread apart exponentially between the numerical solution and the analytically bounded solution.

For any set of the parameters η, α, k and the initial and boundary constants x_0, t_0, $\varphi^{(0)}(x_0,t_0)$, equations (20) describe the $\rho-k-\omega$ curves in the parameter space for $\rho>0$, $k>0$ and $\omega>0$. We can easily show that the distribution of the boundedness curves sensitively depends on the initial and boundary constant $\varphi^{(0)}(x_0,t_0)$ as follows [19,20]. Setting $F(\varphi^{(0)}) = \ln\tan[\varphi^{(0)}(x_0,t_0)/4]$, then $F(\varphi^{(0)})$ takes very great value in the neighborhood of $\varphi^{(0)}(x_0,t_0) = 2\pi$ and any small change of $\varphi^{(0)}(x_0,t_0)$ will lead to large change of the $F(\varphi^{(0)})$. The great $F(\varphi^{(0)})$ value makes the η, α, k curves the quite dense ones. At the point $\varphi^{(0)}(x_0,t_0) = 2\pi$, $F(\varphi^{(0)})$ tends to infinity. This

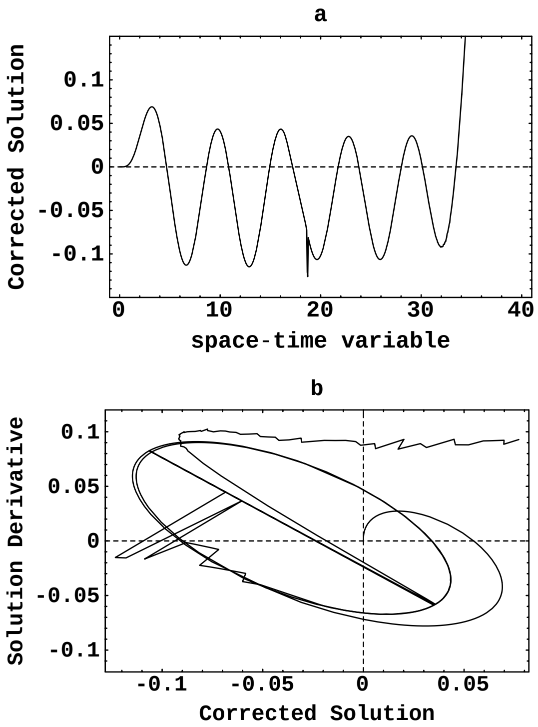

Fig. 1. (a) Plots of the ξ space-time evolution from equation (23) for the parameters $k = \sqrt{2}$, $\omega = 1$, $\eta = \rho = 0.1$, $k\xi_0 = \pi/2$ and $\alpha = 0.025\pi[1 + \mathrm{sech}(\pi/2)]$. (b) The corresponding phase orbit, the first-order corrected solution $\varphi^{(1)}$ *versus* solution derivative $\varphi_\xi^{(1)}$.

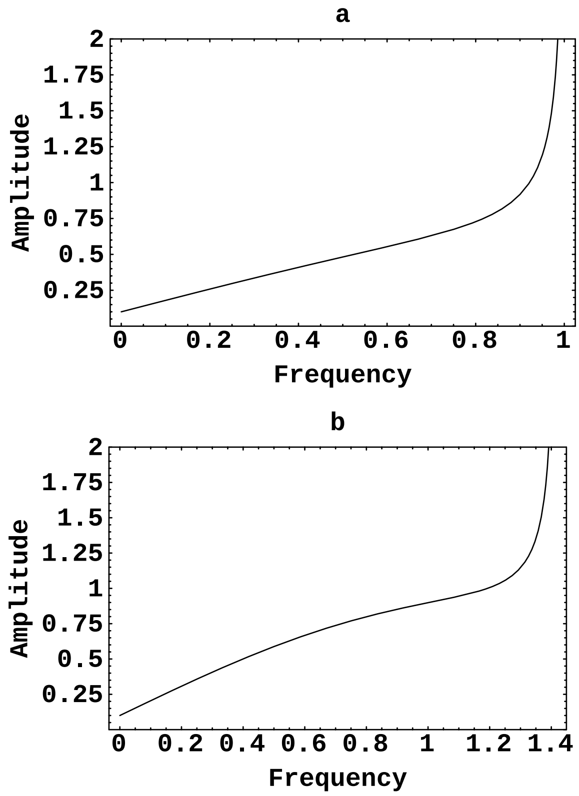

Fig. 2. Chaotic region in parameter space from equation (24). (a) The frequency ω *versus* amplitude ρ for $\eta = 0.1$, $\alpha = 0.25$, $k = 1$; (b) The $\rho - \omega$ plot for $\eta = 0.1$, $\alpha = 0.25$, $k = 2$.

leads to infinitely dense η, α, k curves in a certain region, since the infinite $F(\varphi^{(0)})$ is included in the sinusoidal function of equations (20) as a factor of $\sqrt{k^2 - \omega^2}$. In the case the area ratio between the curves and the white intervals is equal to one. It is clear that the parameters values on the curves are corresponded to the bounded solitons and ones on the white intervals the unbounded solutions. The infinitely dense curves imply that the probability of the boundedness is $1/2$ and it sensitively depends on the initial conditions and system parameters. The result agrees with the previous analysis to the spatially independent chaos [19,20]. In fact, equations (12) are similar to the heteroclinic solutions and equations (20) the Melnikov's chaos criterion [23,19] for the spatially independent function φ. Particularly, equations (20) give the chaos region

$$\rho \geq \left| \frac{2\eta}{\sqrt{k^2 - \omega^2}} \pm \frac{8\alpha\omega}{\pi(k^2 - \omega^2)} \right| \tag{24}$$

in parameter space. From equation (24) with positive sign we make the plot of chaotic region in parameter space as Figure 2: (a) The frequency ω *vs.* amplitude ρ for $\eta = 0.1$, $\alpha = 0.25$, $k = 1$; (b) The $\rho - \omega$ plot for $\eta = 0.1$, $\alpha = 0.25$, $k = 2$. Figure 2 shows that the chaotic region is similar to that of the heteroclinic chaos [23]. For any set of the initial and boundary conditions, we can adjust the control parameters ρ, ω and k in chaotic region to fit the boundedness condition (20), and produce the bounded and chaotic solitons with velocity given by equation (19). The soliton velocity is an important quantity in physics. For example, the emitted power from the long Josephson junction is proportional to [12] $D^2(1-D^2)^{-3/2}$. From equations (19) and (20) we can set $\alpha \approx 0$ and $\omega \approx k$ such that the velocity D approaches 1 and the emitted power reaches very great. The further investigation on applications of the chaotic solitons will be important and interesting.

We have extended a method for solving the perturbed Sine-Gordon equations by using idea from the constant-variation of differential equations. The method can be directly applied to φ^4 field and other nonlinear scalar field. Employing this to the periodically driven long Josephson junction, we have demonstrated existence of the chaotic kink and antikink solitons and shown their several important properties. Perturbed solutions of the chaotic solitons are expressed by equation (8) and the integrations (11) and (14) with a few insolvable terms. The necessary and sufficient boundedness conditions of the chaotic solitons are given as the relationships (19) and (20) among the system parameters and initial and boundary constants, which lead to the chaotic region (24) being similar to that of the heteroclinic chaos. It has been numerically shown that one can produce the chaotic solitons by adjusting the control parameters to fit equations (20) and obtain required soliton velocity through equation (19). We think that the chaotic solitons collecting the regularity and randomness would be doubly useful in some physical applications.

This work was supported by the National Natural Science Foundation of China under Grant No.19874019, and by the Science Foundation of Hunan Provincial Education Commission of China.

References

1. P. Berge, Y. Pomeau, C. Vidal, *Order Within Chaos* (Wiley, New York, 1984).
2. P.K. Bullough, P.J. Caudrey, *Solitons* (Springer, Berlin, 1980).
3. M.C. Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer-Verlag, New York, 1990).
4. F.Kh. Abdullaev, Phys. Rep. **179**, 1 (1989).
5. F.M. Mitschke, L.F. Mollenauer, Opt. Lett. **12**, 355 (1987).
6. W.S. Ni *et al.*, Pro. Phys. **16**, 645 (1996).
7. G.D. VanWiggeren, R. Roy, Phys. Rev. Lett. **81**, 3547 (1998).
8. K. Nozaki, Phys. Rev. Lett. **49**, 1883 (1982).
9. J.C. Eilbeck, P.S. Lomdahl, A.C. Newell, Phys. Lett. A **87**, 1 (1981).
10. K. Nozaki, N. Bekki, Phys. Rev. Lett. **50**, 1226 (1983).
11. B.D. Josephson, Adv. Phys. **14**, 419 (1965).
12. N.G. Jensen, J.A. Blackburn, Phys. Rev. Lett. **70**, 1251 (1993).
13. M.J. Rice *et al.*, Phys. Rev. Lett. **36**, 432 (1976).
14. G.S. Edwards *et al.*, Phys. Rev. Lett. **53**, 1284 (1984).
15. C.T. Zhang, Phys. Rev. A **35**, 886 (1987); **40**, 2148 (1989).
16. W. Hai, Phys. Lett. A **186**, 309 (1994).
17. W. Hai, Y. Xiao, Acta Phys. Sinica (Overseas Edition) **5**, 756 (1996).
18. W. Hai *et al.*, Commu. Theor. Phys. **29**, 627 (1998).
19. W. Hai *et al.*, J. Phys. A **31**, 2991 (1998); Phys. Lett. A **265**, 128 (2000).
20. W. Hai *et al.*, Phys. Lett. A **275**, 54 (2000).
21. J. Zagrodzinski, J. Phys. A **17**, 3315 (1984).
22. B.Z. Liu, *Nonlinear Dynamics and Chaos* (East-North Normal Univ. Press., Changchun, 1994) (in Chinese).
23. M. Bartuccelli *et al.*, Phys. Rev. B **33**, 4686 (1986).